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Note

Reducibility classes of P-selective sets ☆

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Abstract

A set is P-selective (Selman, 1979) if there is a polynomial-time semidecision algorithm for the set – an algorithm that given any two strings decides which is "more likely" to be in the set. This paper establishes a strict hierarchy among the various reductions and equivalences to P-selective sets.

1. Introduction

Given the large number of important problems that do not appear to have easy solutions, researchers have explored more flexible approaches to efficient set recognition (or near-recognition): almost polynomial time [14], average polynomial time (see [6]), implicit membership testability [7], near-testability [5], P-closeness [16, 20], P-selectivity [17], and others. One such notion, that of the P-selective sets, has proven useful in many contexts, such as characterizing P [4] and understanding whether SAT may have unique solutions [10]. Intuitively, a set is P-selective if there is a 2-ary polynomial-time function that chooses which of its inputs is "more likely" (or, actually, "not less likely") to belong to the set.

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Definition 1.1 (Selmen [17]). A set A is P-selective if there is a total polynomial-time function f so that, for every $x, y \in \Sigma^*$,
1. $f(x, y) \in \{x, y\}$, and
2. if $x \in A$ or $y \in A$, then $f(x, y) \in A$.

We will use P-sel to denote the class of P-selective sets. The function f is said to be a *selector* (function) for A.

Over the years, P-selectivity has attracted a good deal of interest and, recently, there have been many advances in our understanding of this class [19, 10, 9, 4, 3, 15, 1, 2]. Of particular interest has been the question of whether NP-complete problems are reducible to P-selective sets – or, more specifically, whether such reducibility implies P = NP. The analog of this question in the theory of sparse sets, arising from the Berman–Hartmanis conjecture, has been intensely investigated over the past decade (see the survey [21]). Ogihara [15], Agrawal and Arvind [1], and Beigel et al. [2] have independently proven that no NP-complete set is $\leqslant^{\mathrm{p}}_{\mathrm{btt}}$-reducible (or even sublinear-tt-reducible [15, 2]) to any P-selective set unless P = NP. A natural question arising from this is whether a similar result holds for more flexible reducibilities such as $\leqslant^{\mathrm{p}}_{\mathrm{tt}}$. Of course, if $\leqslant^{\mathrm{p}}_{\mathrm{btt}}$ reductions to P-selective sets and $\leqslant^{\mathrm{p}}_{\mathrm{tt}}$ reductions to P-selective sets were shown to yield the same class, then the results just mentioned would implicitly handle the $\leqslant^{\mathrm{p}}_{\mathrm{tt}}$ case. Determining whether this is so provides a motivation for this paper in addition to the motivation of seeking to understand the interplay of polynomial-type reductions with central classes such as the P-selective sets. This paper studies reducibility and equivalence to P-selective sets, and proves tight collapse and separation results. Since $\leqslant^{\mathrm{p}}_{\mathrm{T}}$-reducibility to P-selective sets coincides with P/poly (see, e.g., [17]), this paper provides a classification of the sets between P and P/poly. We note that, though our techniques bear no relation to the techniques used to study sparse sets, reductions and equivalences to *sparse* sets and *tally* sets have been satisfyingly studied in a long line of research (see the survey [8]).

For a reducibility $\leqslant^{\mathrm{p}}_{\mathrm{r}}$, let $\mathrm{R_r}$(P-sel) denote the class of sets that are $\leqslant^{\mathrm{p}}_{\mathrm{r}}$-reducible to some P-selective set and let $\mathrm{E_r}$(P-sel) denote the class of sets that are $\leqslant^{\mathrm{p}}_{\mathrm{r}}$-equivalent to some P-selective set. The following is a summary of known results and the results proven in this paper; we refer the reader to [13, 18] for definitions of the standard reducibility notions.

1. [18, 3]

$$\mathrm{P} \subsetneq \text{P-sel} = \mathrm{R_m}(\text{P-sel}) = \mathrm{R_{ptt}}(\text{P-sel}) = \mathrm{R_{pos}}(\text{P-sel}) = \mathrm{E_m}(\text{P-sel}) \subsetneq \mathrm{E_{tt}}(\text{P-sel}).$$

2. ([17] [18, Theorem 12] [12, Theorem 3]) $\mathrm{R_T}$(P-sel) = P/poly.
3. (Watanabe, referenced in [19]) $\mathrm{R_{tt}}(\text{P-sel}) \subsetneq \mathrm{R_T}(\text{P-sel})$.

Note that (3) also follows from the fact that every polynomial-time membership comparable set is in P/poly and some are not in $\mathrm{R_{tt}}$(P-sel) [15].

4. (By Propositions 2.3 and 3.1)

$$\text{P-sel} \subsetneq \mathrm{E_{1\text{-}tt}}(\text{P-sel}) = \mathrm{E_{1\text{-}T}}(\text{P-sel}) = \mathrm{R_{1\text{-}tt}}(\text{P-sel}) = \mathrm{R_{1\text{-}T}}(\text{P-sel}).$$

Table 1

	R_{tt}	R_T	E_{tt}	E_T
$R_{k\text{-tt}}$	Exactly k	Exactly $\lceil \log_2(k+1) \rceil$	Does not include	Does not include
$R_{k\text{-T}}$	Exactly $2^k - 1$	Exactly k	Does not include	Does not include
$E_{k\text{-tt}}$	Exactly k	Exactly $\lceil \log_2(k+1) \rceil$	Exactly k	Exactly k
$E_{k\text{-T}}$	At most $2^k - 1$	Exactly k	At most $2^k - 1$	Exactly k

5. For every $k \geqslant 2$, it holds that

$$\begin{aligned} R_{k\text{-T}}(\text{P-sel}) &= R_{(2^k-1)\text{-tt}}(\text{P-sel}) && \text{(by Proposition 2.2)}, \\ R_{(k-1)\text{-T}}(\text{P-sel}) &\subsetneq R_{k\text{-T}}(\text{P-sel}) && \text{(by Corollary 3.6)}, \\ R_{(k-1)\text{-tt}}(\text{P-sel}) &\subsetneq R_{k\text{-tt}}(\text{P-sel}). && \text{(by Corollary 3.6)}. \end{aligned}$$

6. For every $k \geqslant 2$, it holds that

$$\begin{aligned} E_{k\text{-T}}(\text{P-sel}) &\subsetneq E_{(2^k-1)\text{-tt}}(\text{P-sel}) && \text{(by Corollary 3.11)}, \\ E_{(k-1)\text{-T}}(\text{P-sel}) &\subsetneq E_{k\text{-T}}(\text{P-sel}) && \text{(by Corollary 3.6)}, \\ E_{(k-1)\text{-tt}}(\text{P-sel}) &\subsetneq E_{k\text{-tt}}(\text{P-sel}) && \text{(by Corollary 3.6)}, \\ E_{k\text{-tt}}(\text{P-sel}) &\not\subseteq E_{(k-1)\text{-T}}(\text{P-sel}) && \text{(by Theorem 3.10)}. \end{aligned}$$

7. (By Corollary 3.3) For every $k \geqslant 2$,

$$E_{k\text{-tt}}(\text{P-sel}) \subsetneq R_{k\text{-tt}}(\text{P-sel}) \quad \text{and} \quad E_{k\text{-T}}(\text{P-sel}) \subsetneq R_{k\text{-T}}(\text{P-sel}).$$

8. (By Corollary 3.9) R_{btt}(P-sel) $\subsetneq$ R_{tt}(P-sel) $\subsetneq$ R_T(P-sel), E_{btt}(P-sel) $\subsetneq$ E_{tt}(P-sel) $\subsetneq$ E_T(P-sel), E_{btt}(P-sel) $\subsetneq$ R_{btt}(P-sel), E_{tt}(P-sel) $\subsetneq$ R_{tt}(P-sel), and E_T(P-sel) $\subsetneq$ R_T(P-sel)

9. (By Corollary 3.9) E_{tt}(P-sel) and R_{btt}(P-sel) are incomparable. E_T(P-sel) and R_{btt}(P-sel) are incomparable. E_T(P-sel) and R_{tt}(P-sel) are incomparable.

Table 1 shows, for each reduction or equivalence type in the first row, how many queries to some P-selective set are necessary in order to include the k-query ($k \geqslant 2$) classes in the first column. For example, the R_{tt} column in the $R_{k\text{-T}}$ row states that $R_{k\text{-T}}(\text{P-sel}) = R_{(2^k-1)\text{-tt}}(\text{P-sel})$.

An interesting open question is whether the $2^k - 1$ upper bounds in the last row are tight; i.e., whether $E_{k\text{-T}}(\text{P-sel}) \subseteq R_{(2^k-2)\text{-tt}}(\text{P-sel})$ and whether $E_{k\text{-T}}(\text{P-sel}) \subseteq E_{(2^k-2)\text{-tt}}(\text{P-sel})$.

2. Preliminaries

For a set A and a natural number n, let $A^{(\leqslant n)} = \{x \in A \mid |x| \leqslant n\}$ and $A^{(n)} = \{x \in A \mid |x| = n\}$. By convention, $\Sigma^{\leqslant n} = (\Sigma^*)^{(\leqslant n)}$ and $\Sigma^n = (\Sigma^*)^{(n)}$.

We say that an arity two function f is a *partial selector on (finite) set* $Q \subseteq \Sigma^*$ if for every $x, y \in Q$, it holds that $f(x,y) \in \{x,y\}$. Given a finite set Q and a partial selector f on Q, define a directed graph $G_{f,Q} = (Q, A)$ as follows:

$(u, v) \in A$ if and only if either $f(u, v) = v$ or $f(v, u) = v$.

We say that u and $v \in Q$ are *equivalent with respect to* f *in* Q if there is a loop in $G_{f,Q}$ containing both u and v; namely, u and v are reachable from one another in $G_{f,Q}$. Obviously, the equivalence relation is reflexive, symmetric, and transitive.

Definition 2.1. Let f be a partial selector on a finite set Q and let $Q_1, \ldots, Q_m$ be nonempty subsets of Q. We say that $[Q_1, \ldots, Q_m]$ is a *decomposition of* Q *with respect to* f if the following conditions are satisfied:

1. $Q_1 \bigcup \cdots \bigcup Q_m = Q$,
2. for every i, j, $1 \leqslant i < j \leqslant m$, it holds that $Q_i \bigcap Q_j = \emptyset$,
3. for every i, j, $1 \leqslant i < j \leqslant m$, and for every $u \in Q_i$ and $v \in Q_j$, it holds that $f(u, v) = f(v, u) = v$ and u and v are not equivalent with respect to f in Q, and
4. for every i, $1 \leqslant i \leqslant m$, and for every $u, v \in Q_i$, u and v are equivalent with respect to f in Q.

McLaughlin and Martin, cited in [11], showed that the semirecursive sets – the sets with recursive selector functions – are equivalent to left cuts of binary reals. The following two facts, which are easy to verify, are analogous to the mentioned result.

Fact 1. *Let f be a partial selector on Q. There exists a unique decomposition of Q with respect to f.*

Fact 2. *Let f be a selector for a set X and let $[Q_1, \ldots, Q_m]$ be the decomposition of a finite set Q with respect to f. Then there exists a unique i, $0 \leqslant i \leqslant m$, such that*

$$Q_1, \ldots, Q_i \subseteq \overline{X} \quad \text{and} \quad Q_{i+1}, \ldots, Q_m \subseteq X.$$

Noting that the problem of computing decomposition is reducible to the reachability problem, we have the following fact.

Fact 3. *Let f be a partial selector on Q with $||Q|| = m$ and $Q \subseteq \Sigma^{\leqslant n}$. Suppose $f(x, y)$ is computable in time $(|x| + |y|)^c$. Then the decomposition of Q with respect to f is computable in time polynomial in $m \cdot n^c$.*

Throughout this paper, $M_1, M_2, \ldots$ is a standard enumeration of polynomial-time deterministic oracle Turing machines, where M_i runs, independent of its oracle, in time $n^i + i$ on inputs of length n. $f_1, f_2, \ldots$ is a standard enumeration of all polynomial-time computable 2-ary functions. We assume that $f_i(x, y)$ is computable in time $(|x|+|y|)^i+i$ for every x and y. Define $\mu(n)$ by: $\mu(0) = 2^{2^2}$, and $\mu(n) = 2^{2^{2^{\mu(n-1)}}}$ for $n \geqslant 1$.

Let U be a finite subset of Σ^*. Let $x_1, \ldots, x_m$ be the enumeration of all elements in U in increasing lexicographic order. For $A \subseteq U$ and $B \subseteq U$, we say that A is *smaller* than B if:

$$\chi_A(x_1)\cdots\chi_A(x_m) \text{ is lexicographically smaller than } \chi_B(x_1)\cdots\chi_B(x_m),$$

where $\chi_A(x)=1$ ($\chi_B(x)=1$) if $x\in A$ ($x\in B$) and equals 0 (0) otherwise. By introducing this ordering, for any finite subset U of Σ^* it holds that 2^U, the power set of U, is totally ordered.

We now state a combinatorial fact, which will be used later.

Fact 4. *For every $k\geqslant 2$, there exists a mapping g of $\{1,\ldots,k^2\}$ to $\{1,\ldots,k\}$ such that the following conditions are all satisfied:*

(c1) *For every $l, 1\leqslant l\leqslant k$, $\|g^{-1}(l)\| = k$.*

(c2) *For every $u, k^2-k+2\leqslant u\leqslant k^2$, $g(u)=k$.*

(c3) *For each $u, 0\leqslant u\leqslant k^2$ and $l, 1\leqslant l\leqslant k$, let $\gamma_{\leqslant u}(l)$ denote the parity of $\|\{v \mid 1\leqslant v\leqslant u \text{ and } g(v) = l\}\|$, and for each $u, 0\leqslant u\leqslant k^2$, let $\Gamma_{\leqslant u}$ denote $(\gamma_{\leqslant u}(1),\ldots, \gamma_{\leqslant u}(k))$. Then $\Gamma_{\leqslant 0},\ldots,\Gamma_{\leqslant k^2-k+1}$ are all distinct.*

The following proposition follows from Fact 2.

Proposition 2.2. *For any $k\geqslant 1$,* $\mathrm{R}_{k\text{-T}}(\text{P-sel}) = \mathrm{R}_{(2^k-1)\text{-tt}}(\text{P-sel})$.

We show that all the one-query classes are the same.

Proposition 2.3. $\mathrm{R}_{1\text{-T}}(\text{P-sel}) = \mathrm{R}_{1\text{-tt}}(\text{P-sel}) = \mathrm{E}_{1\text{-T}}(\text{P-sel}) = \mathrm{E}_{1\text{-tt}}(\text{P-sel})$.

Proof. It suffices to show that $\mathrm{R}_{1\text{-tt}}(\text{P-sel})\subseteq \mathrm{E}_{1\text{-tt}}(\text{P-sel})$. Let L be $\leqslant^{\mathrm{p}}_{1\text{-tt}}$-reducible to a P-selective set A via a machine M. Define $B = \{\langle x,y\rangle \mid M$ on x queries y, the acceptance of $M(x)$ depends on the oracle answer to y, and $y\in A\}$. Clearly, B is P-selective and $L\leqslant^{\mathrm{p}}_{1\text{-tt}}B$. Consider a machine D that, given $w = \langle x,y\rangle$, simulates M on x and, behaves as follows:

(1) if $M(x)$ does not query y, then D rejects w,

(2) if $M(x)$ queries y, but ignores the answer to y, then D rejects w,

(3) if $M(x)$ queries y and accepts x iff the answer is YES, then D accepts w if and only if $x\in L$, and

(4) if $M(x)$ queries y and accepts x iff the answer is NO, then D accepts w if and only if $x\notin L$.

It is not hard to see that D is a $\leqslant^{\mathrm{p}}_{1\text{-tt}}$ reduction of B to L. □

3. Separation results

We first separate $\mathrm{R}_{1\text{-tt}}(\text{P-sel})$ from P-sel.

Proposition 3.1. $\text{P-sel} \subsetneq \mathrm{R}_{1\text{-tt}}(\text{P-sel})$.

Proof. We will construct an infinite binary string $w = w_1w_2w_3\cdots$. Let *left*(w) be the set of all finite strings x that are less than w in the dictionary order and define $A = \{\langle b,x\rangle \mid \text{either } (b = 0 \text{ and } x \notin \mathit{left}(w)) \text{ or } (b = 1 \text{ and } x \in \mathit{left}(w))\}$. As the dictionary order is total, *left*(w) is P-selective. Clearly, A is $\leqslant^{\mathrm{p}}_{1\text{-tt}}$-reducible to *left*$(w)$. The bits of w are constructed in blocks of four consecutive bits, whose first and second bits are fixed as 0 and 1, respectively. The third and the fourth bits are determined as follows. Suppose we are to determine the third and fourth bits of the ith block, and let u be w_1w_2 if $i = 1$ and $w_1\cdots w_{4(i-1)}w_{4i-3}w_{4i-2}$ otherwise. We simulate f_i on input $(\langle 0,u10\rangle,\langle 1,u01\rangle)$. If f_i outputs its first argument, then we set $w_{4i-1}w_{4i} = 11$, and otherwise we set $w_{4i-1}w_{4i} = 00$, which establishes that f_i is not a selector function for A. □

Our separations of reducibility classes from equivalence classes follow from the following theorem.

Theorem 3.2. $\mathrm{R}_{2\text{-tt}}(\text{P-sel}) \not\subseteq \mathrm{E}_{\mathrm{T}}(\text{P-sel})$.

Proof. We construct in stages a set S, which contains at most one string of each length and contains only strings of length $\mu(s)$ for some s. Each stage s determines the membership of strings of length $n = \mu(s)$ and our construction is designed so that for every s, all strings that are put into S prior to stage s can be enumerated in time polynomial in $\mu(s+1)$.

At stage $s = \langle i,j,l\rangle$, we will diagonalize against a pair of machines (M_i,M_j) and a function f_l, to establish the following requirement (R):

(R) for any set X with selector f_l, either $S \leqslant^{\mathrm{p}}_{\mathrm{T}} X$ via M_i or $X \not\leqslant^{\mathrm{p}}_{\mathrm{T}} S$ via M_j.

Our construction proceeds as follows. Let $n = \mu(s)$ and let S' be the set of all strings put into S prior to stage s. Let Q be the set of all possible query strings of M_i on x for all $x \in \Sigma^n$. Note that any element in Q is of length at most $n^i + i$ and that $\|Q\| \leqslant 2^n 2^{n^i+i} \leqslant 2^{2n^i}$.

First we check whether

(a) f_l is a partial selector on Q.

If this does not hold, then clearly (R) is satisfied. So we proceed to the next stage without adding any string to S.

Suppose that f_l is a partial selector on Q. Let $[Q_1,\ldots,Q_m]$ be the decomposition of Q with respect to f_l. We check whether the following conditions, (b) and (c), are satisfied.

(b) There exists some t such that

$$Q_1,\ldots,Q_t \subseteq \overline{L(M_j,S')} \text{ and } Q_{t+1},\ldots,Q_m \subseteq L(M_j,S').$$

(c) For every $x \in \Sigma^n$, there exists some $t(x)$ such that

(c1) $Q_1,\ldots,Q_{t(x)} \subseteq \overline{L(M_j,S' \cup \{x\})}$ and

(c2) $Q_{t(x)+1},\ldots,Q_m \subseteq L(M_j,S' \cup \{x\})$.

If (b) does not hold, we add nothing new to S. If (b) does hold and (c) does not hold, we pick the smallest x not satisfying (c) and put it into S. By doing this, we establish that for every X with selector f_l, $X \not\leqslant_{\mathrm{T}}^{\mathrm{p}} S$ via M_j, as there exist $a, b \in Q$ such that (i) $M_j^S(a)$ accepts, (ii) $M_j^S(b)$ rejects, and (iii) $a \in X$ implies $b \in X$. Thus (R) is satisfied. We proceed to the next stage.

Now suppose that (b) and (c) are satisfied. Note for any $x \in \Sigma^n$, that $t(x)$ in condition (c) is uniquely determined. Then there exist $x, y \in \Sigma^n$ such that $x < y$ and $t(x) = t(y)$. This is seen as follows. Assume otherwise, i.e., that all $t(x)$ are distinct. Let r be the (2^{n-1})th largest $t(x)$ and let $U = \{x \mid t(x) \geqslant r\}$ and let $V = \{x \mid t(x) < r\}$. Let $q \in Q_r$. Then $M_j^{S' \cup \{x\}}(q)$ rejects for every $x \in U$ and $M_j^{S' \cup \{x\}}(q)$ accepts for every $x \in V$. Suppose that $M_j^{S'}(q)$ accepts. Since $M_j^{S' \cup \{x\}}(q)$ rejects for every $x \in U$, every $x \in U$ is queried by $M_j^{S'}(q)$. On the other hand, suppose that $M_j^{S'}(q)$ rejects. Since $M_j^{S' \cup \{x\}}(q)$ accepts for every $x \in V$, every $x \in V$ is queried by $M_j^{S'}(q)$. So, either every $x \in U$ is queried by $M_j^{S'}$ on q or every $x \in V$ is queried by $M_j^{S'}$ on q. Since deterministic machine M_j is $q_j(n)$ time bounded, the number of queries of $M_j^{S'}$ on q is at most $|q|^j + j \leqslant (n^i + i)^j + j < 2^{n-1} - 1$. Thus, since $\|U\| = \|V\| = 2^{n-1}$, we have a contradiction.

Now let (u, v) be the smallest pair (x, y) such that $x < y$ and $t(x) = t(y)$. For every $q \in Q$, it holds that

$$M_j^{S' \cup \{u\}}(q) \text{ accepts if and only if } M_j^{S' \cup \{v\}}(q) \text{ accepts.}$$

So M_i relative to $L(M_j, S' \cup \{u\})$ and M_i relative to $L(M_j, S' \cup \{v\})$ accept the same language. We put u into S if v is in the language and v into S otherwise. Then either (i) $v \notin S$ and $v \in L(M_i, L(M_j, S))$, or (ii) $v \in S$ and $v \notin L(M_i, L(M_j, S))$. Thus, (R) is satisfied and we proceed to the next stage.

It remains to show that S is $\leqslant_{2\text{-tt}}^{\mathrm{p}}$-reducible to some P-selective set. Define A to be the set of all strings x for which there is some $y \in S$ such that $x \leqslant y$ and $|x| = |y|$. Clearly, for every x, $x \in S$ if and only if $x \in A$ and $x' \notin A$, where x' is successor of x. So $S \leqslant_{2\text{-tt}}^{\mathrm{p}} A$. In order to prove that A is P-selective, notice that whether $\ell = \mu(s)$ holds can be tested in time polynomial in ℓ. Given two strings x and y, if either of x or y is not of length $\mu(s)$ for any s, then such a string certainly belongs to $\overline{A}$. If both of them are of the same length and the length is $\mu(s)$ for some s, then the smaller one is more likely to belong to A than the larger one. If both of them are of length $\mu(s)$ for some s but are not of the same length, then the membership of the smaller string can be determined in time polynomial in the length of the longer string. Thus, A is P-selective. This proves the theorem. □

Corollary 3.3. *For any* $k \geqslant 2$, $\mathrm{E}_{k\text{-tt}}(\text{P-sel}) \subsetneq \mathrm{R}_{k\text{-tt}}(\text{P-sel})$ *and* $\mathrm{E}_{k\text{-T}}(\text{P-sel}) \subsetneq \mathrm{R}_{k\text{-T}}(\text{P-sel})$.

Corollary 3.4. $\mathrm{E}_{\mathrm{btt}}(\text{P-sel}) \subsetneq \mathrm{R}_{\mathrm{btt}}(\text{P-sel})$, $\mathrm{E}_{\mathrm{tt}}(\text{P-sel}) \subsetneq \mathrm{R}_{\mathrm{tt}}(\text{P-sel})$, *and* $\mathrm{E}_{\mathrm{T}}(\text{P-sel}) \subsetneq \mathrm{R}_{\mathrm{T}}(\text{P-sel})$.

We next show that both reducibility classes and equivalence classes have proper hierarchies, according to the (constant) number of queries.[4]

Theorem 3.5. *For each* $k \geqslant 2$, $\mathrm{E}_{k\text{-tt}}(\text{P-sel}) \not\subseteq \mathrm{R}_{(k-1)\text{-tt}}(\text{P-sel})$.

Proof. Let $k \geqslant 2$. We will construct a set T and a P-selective set A in stages. At stage $s = \langle i,j \rangle$, we will diagonalize against machine M_i and function f_j so that for any set X with selector f_j, $T \not\leqslant^{\mathrm{p}}_{(k-1)\text{-tt}} X$ via M_i.

Both T and A will consist only of strings of length $k\mu(s)$ for some s and $A^{(k\mu(s))}$ will be an initial segment of $\Sigma^{k\mu(s)}$ of size at most k^2.

Let T' be the set of all strings put into T prior to stage s and let $n = k\mu(s)$. Let $x_1,\dots,x_k$ be the smallest (in this order) k strings of length n. We check whether M_i behaves as a $k-1$ truth-table reduction for any inputs in $\{x_1,\dots,x_k\}$ and if so, f_j behaves as a partial selector on Q, where Q is the union of the query strings of M_i on $x_t, 1 \leqslant t \leqslant k$. If the above check fails, then we proceed to the next stage without adding any new elements to T.

Let $[Q_1,\dots,Q_m]$ be the decomposition of Q with respect to f_j. Since $\|Q\| \leqslant k(k-1)$, $m \leqslant k(k-1)$. By Fact 2, for every X with selector f_j, there is some t such that $Q_1,\dots,Q_t \subseteq \overline{X}$ and $Q_{t+1},\dots,Q_m \subseteq X$. For t, $0 \leqslant t \leqslant m$, and l, $1 \leqslant l \leqslant k$, let $\tau_t(l)$ be 1 if $M_i(x_l)$ accepts relative to $T' \cup (Q_{t+1} \cup \cdots \cup Q_m)$ and 0 otherwise, and let $\tau_t = (\tau_t(1),\cdots,\tau_t(k))$. Let g be the function in Fact 4 for k. Since $0 \leqslant m \leqslant k^2-k$ and (using the notation of Fact 4) $\Gamma_{\leqslant u} \neq \Gamma_{\leqslant v}$ for every u and v, $0 \leqslant u < v \leqslant k^2-k+1$, there is at least one u such that $\Gamma_{\leqslant u} \neq \tau_t$ for every t. Let w be the smallest such u. For each $l, 1 \leqslant l \leqslant k$, we add x_l to T if and only if (again using the notation of Fact 4) $\gamma_{\leqslant w}(l) = 1$, and we add to A $x_1,\dots,x_w$, the smallest w strings of length $k\mu(s)$. Then, for any set X with selector f_j, there is some l such that $x_l \in T$ if and only if $M_i^X(x_l)$ rejects. So (R) is satisfied. We proceed to the next stage.

We claim that A and T are $\leqslant^{\mathrm{p}}_{k\text{-tt}}$ equivalent. Note for any l, that

$$
\begin{aligned}
x_l \in T &\Leftrightarrow \gamma_{\leqslant w}(l) = 1 \\
&\Leftrightarrow \|\{v \mid v \leqslant w, g(v) = l\}\| \text{ is odd} \\
&\Leftrightarrow \|\{v \mid 1 \leqslant v \leqslant k^2, g(v) = l, v \leqslant w\}\| \text{ is odd} \\
&\Leftrightarrow \|\{v \mid 1 \leqslant v \leqslant k^2, g(v) = l, x_v \in A\}\| \text{ is odd.}
\end{aligned}
$$

Let B_l be the set of all v such that $g(v) = l$. Then $x_l \in T$ if and only if $\|B_l \bigcap A\|$ is odd. By definition, $\|B_l\| = k$. Since B_l is easily computable, we have $T \leqslant^{\mathrm{p}}_{k\text{-tt}} A$. On the other hand, it holds that $\Gamma^w_g = (\chi_T(x_1),\dots,\chi_T(x_k))$. So, by Fact 4, w can be computed from $\chi_T(x_1),\dots,\chi_T(x_k)$. For any l, $x_l \in A$ if and only if $l \leqslant w$. So, $A \leqslant^{\mathrm{p}}_{k\text{-tt}} T$. It is not hard to see that A is P-selective. This proves the theorem. $\square$

[4] Note added in proof: Burtschick and Lindner ("On Sets Turing Reducible to P-Selective Sets", University of Ulm, Department of Computer Science Technical Report 95-09, August 1995) have recently studied the analogous non-constant case.

Corollary 3.6. $R_{(k-1)\text{-tt}}(\text{P-sel}) \subsetneq R_{k\text{-tt}}(\text{P-sel})$, $E_{(k-1)\text{-tt}}(\text{P-sel}) \subsetneq E_{k\text{-tt}}(\text{P-sel})$, *and* $R_{(k-1)-\text{T}}$ P-sel $\subsetneq R_{k\text{-T}}(\text{P-sel})$.

As to unbounded query classes, we have the following theorem.

Theorem 3.7. $E_{\text{T}}(\text{P-sel}) \not\subseteq R_{\text{tt}}(\text{P-sel})$.

Proof. We give a sketch of the proof. We will construct a set L that is $\leqslant_{\text{T}}^{\text{p}}$-equivalent to a P-selective set A, but not $\leqslant_{\text{tt}}^{\text{p}}$-reducible to any P-selective set. For any n, either L or A contains a string of length n only if $n = \mu(s)$ for some s. Let $n = \mu(s)$ and $s = \langle i,j \rangle$. Let $x_1,\ldots,x_n$ be the n smallest (in this order) length n strings. Then $L^{(n)} \subseteq \{x_1,\ldots,x_n\}$ and $A^{(n)}$ will be a nonempty initial segment of Σ^n. Let the segment be $[0^n, w]$. Then, for each t, $1 \leqslant t \leqslant n$, $x_t \in L$ if and only if the tth bit of w is a 1. The string w is set to 0^n either if M_i does not behave as a truth-table reduction for some input $x_t, 1 \leqslant t \leqslant n$, or M_i does behave as such a reduction but F_j does not behave as a partial selector on Q, where Q is the union of all query strings of M_i on x_t for some $t, 1 \leqslant t \leqslant n$. Otherwise, we compute the decomposition $[Q_1,\ldots,Q_m]$ of Q. Clearly, $m \leqslant n \cdot p_i(n) < 2^{p_i(n)}$. So, it is possible to assign membership to the strings $x_1,\ldots,x_n$, so that L cannot be reduced to a set X with partial selector F_j via M_i.

Note that, in order to test the membership of x_t, it suffices to compute w by a binary search method and that, in order to test whether $u \leqslant w$, it suffices to compute bits of w by queries to L. So, L and A are $\leqslant_{\text{T}}^{\text{p}}$-equivalent. The P-selectivity of A follows from a discussion similar to that of Theorem 3.2. This proves the theorem. □

In the above proof, by shrinking the possible size of the initial segment from 2^n to n and diagonalizing against all $\log^* n$ truth-table reductions, we get the following theorem.

Theorem 3.8. $E_{\text{tt}}(\text{P-sel}) \not\subseteq R_{\text{btt}}(\text{P-sel})$.

Corollary 3.9. $R_{\text{btt}}(\text{P-sel}) \subsetneq R_{\text{tt}}(\text{P-sel}) \subsetneq R_{\text{T}}(\text{P-sel})$, $E_{\text{btt}}(\text{P-sel}) \subsetneq E_{\text{tt}}(\text{P-sel}) \subsetneq E_{\text{T}}(\text{P-sel})$, $E_{\text{btt}}(\text{P-sel}) \subsetneq R_{\text{btt}}(\text{P-sel})$, $E_{\text{tt}}(\text{P-sel}) \subsetneq R_{\text{tt}}(\text{P-sel})$, *and* $E_{\text{T}}(\text{P-sel}) \subsetneq R_{\text{T}}(\text{P-sel})$. *Also,* $E_{\text{tt}}(\text{P-sel})$ *and* $R_{\text{btt}}(\text{P-sel})$ *are incomparable,* $E_{\text{T}}(\text{P-sel})$ *and* $R_{\text{btt}}(\text{P-sel})$ *are incomparable, and* $E_{\text{T}}(\text{P-sel})$ *and* $R_{\text{tt}}(\text{P-sel})$ *are incomparable.*

As to the relationship between truth-table equivalence classes and Turing equivalence classes, we have the following theorem.

Theorem 3.10. *For each* $k \geqslant 2$, $E_{k\text{-tt}}(\text{P-sel}) \not\subseteq E_{(k-1)\text{-T}}(\text{P-sel})$.

Proof. Let $k \geqslant 2$. We will construct a set L and a P-selective set A in stages. At stage $s = \langle i,j,l \rangle$, we will diagonalize against M_i and M_j so that for any set X with selector

f_l, either $T \not\leqslant^{\mathrm{p}}_{(k-1)\text{-T}} X$ via M_i or $X \not\leqslant^{\mathrm{p}}_{(k-1)\text{-T}} T$ via M_j. The construction at stage s proceeds as follows

Let T' be the set of all strings put into T prior to stage s. Let $n = k\mu(s)$ and $x_1,\ldots,x_{k+1}$ be the $k+1$ smallest (in this order) strings of length n.

We first check that (i) M_i behaves as a $k-1$ Turing reduction on input x_p, for every p such that $1 \leqslant p \leqslant k+1$, (ii) f_l behaves as a partial selector on Q, the set of all possible query strings of M_i on inputs $x_1,\ldots,x_{k+1}$, (iii) M_j behaves as a $k-1$ Turing reduction on every input in Q, and (iv) there exist x_p and t_p such that

$$Q_1,\ldots,Q_{t_p} \subseteq \overline{L(M_j, T' \cup \{x_p\})} \text{ and } Q_{t_p+1},\ldots,Q_m \subseteq L(M_j, T' \cup \{x_p\}).$$

If one of the first three conditions fails to hold, then our requirement is already satisfied, so we put x_1 into T. If only (iv) fails to hold for some x_p, then by putting such an x_p into T, our requirement is fulfilled.

Now suppose that the check is passed. Then there exist p and p', $1 \leqslant p < p' \leqslant k+1$, such that $t_p = t_{p'}$. This is seen as follows: Assume, to the contrary, that the values t_p are all distinct. Let d be such that t_d is the largest amongst all t_p and let $q \in Q_{t_d}$. For every $p \neq d$, it holds that $q \in L(M_j, T' \cup \{x_p\}) - L(M_j, T' \cup \{x_d\})$. Each oracle $T' \cup \{x_p\}$ contains only one string in $\{x_1,\ldots,x_{k+1}\}$. So M_j on q relative to T' makes queries to all $x_1,\ldots,x_{k+1}$ except x_d. Thus M_j on q relative to T' makes k queries, a contradiction.

Let p_0 and p_0' be the smallest p and p' such that $p < p'$ and $t_p = t_{p'}$. Let $T_1 = T' \cup \{x_{p_0}\}$ and $T_2 = T' \cup \{x_{p_0'}\}$. Since $t_{p_0} = t_{p_0'}$, $L(M_j, T_1) = L(M_j, T_2)$. Let $X = L(M_j, T_1)$. Clearly, since $T_1 \neq T_2$, either $T_1 \neq L(M_i, X)$ or $T_2 \neq L(M_i, X)$. Thus, either $T_1 \neq L(M_i, L(M_j, T_1))$ or $T_2 \neq L(M_i, L(M_j, T_2))$. If the primary condition holds, set $T = T_1$; otherwise, set $T = T_2$. Then our requirement is satisfied.

Now, define $X = \{x_1,\ldots,x_d\}$ so that x_d is in T and set $A = A \cup X$. The size of the segment $\{x_1,\ldots,x_d\}$ can be computed by checking the membership of all $x_2,\ldots,x_{k+1}$ (x_1 is always in A). So T is $\leqslant^{\mathrm{p}}_{k\text{-tt}}$-reducible to A. On the other hand, x_1 is always in A, and, for every $p \geqslant 2$, x_p is in A if and only if $\{x_p,\ldots,x_{k+1}\} \bigcap T \neq \emptyset$. So $A \leqslant^{\mathrm{p}}_{k\text{-tt}} T$. Thus T and A are $\leqslant^{\mathrm{p}}_{k\text{-tt}}$-equivalent. $\square$

Since $\mathrm{E}_{k\text{-T}}(\text{P-sel}) \subseteq \mathrm{E}_{(2^k-1)\text{-tt}}(\text{P-sel})$, Theorem 3.10 yields the following corollary.

Corollary 3.11. *For each* $k \geqslant 2$, $\mathrm{E}_{k\text{-T}}(\text{P-sel}) \subsetneq \mathrm{E}_{(2^k-1)\text{-tt}}(\text{P-sel})$.

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